10/29 Recitation

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 1 & 1 & 1 \end{bmatrix}$.

   (a) Compute the determinant starting with a cofactor expansion along row 1.

   (b) Compute the determinant starting with a cofactor expansion along column 2.

   (c) Compute the determinant using row reduction

2. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. In 10/24 recitation, you used Julia to find the formula

$$\det A = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}.$$

   Derive this formula using the cofactor expansion along the first row.

3. Recall that if A is invertible, then $A^{-1} = \frac{1}{\det A} C^T$, where C is the cofactor matrix. Let's prove this.

   (a) Show that the $(i, j)-$entry of AC^T is $a_{i1}C_{j1} + \cdots + a_{in}C_{jn}$.

   (b) Show that if $i = j$, then $a_{i1}C_{j1} + \cdots + a_{in}C_{jn} = \det A$. Show that if $i \neq j$, then $a_{i1}C_{j1} + \cdots + a_{in}C_{jn} = 0$ by interpreting this expression as the determinant of a different matrix.

   (c) Show that $AC^T = (\det A)I$ and deduce that $A^{-1} = \frac{1}{\det A} C^T$.

4. True/False. If a statement is true, prove it. If it is false, provide counterexample.

   (a) $\det(A + B) = \det A + \det B$

   (b) The determinant of any projection matrix (other than the identity) is 0.

5. **Looking Ahead:** Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $\lambda \in \mathbb{R}$. Find $\det(A - \lambda I)$ and solve the equation $\det(A - \lambda I) = 0$ for λ.